Dr. Stefano Filipazzi
Dr. Alapan Mukhopadhyay
Léo Navarro Chafloque

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AG II - Schemes and sheaves

Exercises – week 7

Exercise 1. *Finite covers of* $\mathbb{P}^1_{\mathbb{C}}$. Let $n \geq 1$. Consider the self map c_n of $\mathbb{C}$-schemes on $\operatorname{Proj}(\mathbb{C}[x, y]) = \mathbb{P}^1_{\mathbb{C}}$ induced by Proj from the $\mathbb{C}$-algebra map $x \mapsto x^n$ and $y \mapsto y^n$ on $\mathbb{C}[x, y]$.

(1) Compute the preimage by this map of $D_+(x)$ and $D_+(y)$, show it's affine and show that the induced map of rings at global sections $c_n^{-1}(D_+(x)) \to D_+(x)$ (and same with y) is finite.[1]
(2) Compute all the fibers of the map.

Exercise 2. *Cones.* Let S be a graded ring finitely generated in degree 1. We define the *Cone* of $\operatorname{Proj}(S)$[2] to be the $\operatorname{Spec}(S)$. We call $v := V(S_+)$ the vertex of the cone.

(1) Consider $T = \operatorname{Bl}_v(\operatorname{Spec}(S))$. Suppose that S is generated in degree 1. Show that the blow-up algebra (see week 5, exercise 3) is
$$S' = \bigoplus_{n \geq 0} \left(\bigoplus_{k \geq n} S_k \right).$$
The aspect *generated in degree 1* is crucial for the above. We suppose this for the rest of the exercise.
(2) Show that the natural graded map $S \to S'$ induces a morphism $f : T \to \operatorname{Proj}(S)$.
(3) Show that f restricted to the exceptional divisor of the blow-up is an isomorphism.
(4) Let a be an element of degree 1 in S. Show that $f^{-1}(D_+(a)) \cong \operatorname{Spec}(S_{(a)}[t])$.

Remark. Some interpretation. Say $S_0 = k$. Take generators in degree 1 of S as a k-algebra. It gives a closed embedding in $\mathbb{P}^n_k$. We can therefore interpret X as a certain subset of lines in k^{n+1}. The *cone* consists of those lines in $\mathbb{A}^{n+1}_k$. The vertex v correspond to the origin where all lines meet. Recall that X can be seen as the $\mathbb{G}_m$-quotient of $\operatorname{Spec}(S) \setminus v$. Blowing up at the vertex replaces v by all directions going into v inside $\operatorname{Spec}(S)$, *i.e.* X. We will soon see that T is a *line bundle* over X, in fact the one associated to $\mathcal{O}_X(-1)$. The zero section of this line bundle correspond to the exceptional divisor in the above point of view.

[1] A map of rings $A \to B$ is finite if B is a finitely generated A-module. This implies that this self map is a *finite map of schemes* a notion to be introduced in the lecture soon.

[2] Note that the following algebra depends on the algebra S, and so not only on the isomorphism class of $\operatorname{Proj}(S)$.

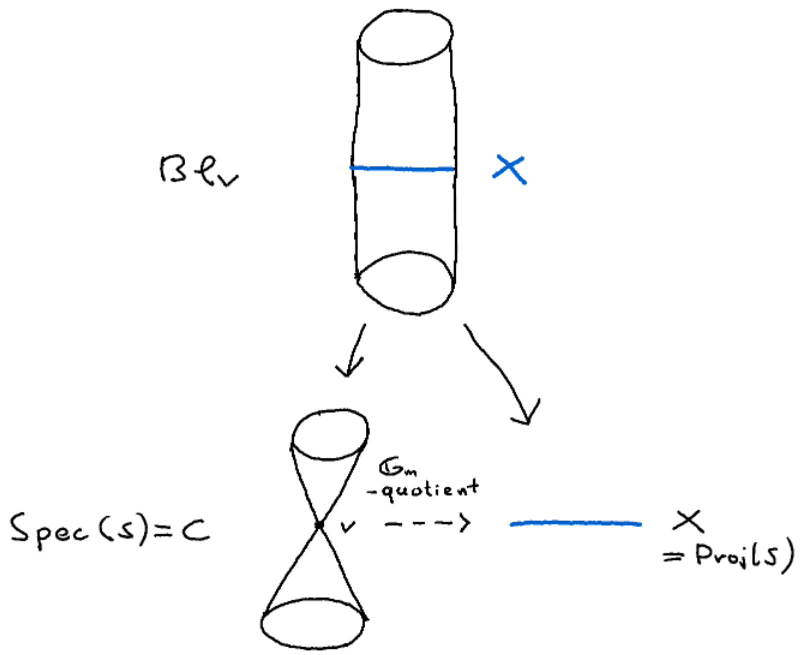

FIGURE 1. Some illustration of the above remark about Exercise 1.

Exercise 3. *Functions on integral schemes and S_2 property.* Let $X = \mathrm{Spec}(A)$ be an integral affine scheme. Denote by η the generic point of X. Denote by $K(X) = \mathcal{O}_{X,\eta}$. Let $f \in K(X)$. Define

$$I = \{g \in A \mid fg \in A\},$$

the *ideals of denominators* of f.

(1) Show that $X \setminus V(I)$ is the largest open subscheme U of X such that $f \in \mathcal{O}_X(U)$.
(2) *Geometric interpretation of the S_2 property.* Suppose that X is an affine integral S_2 scheme. Let $f \in K(X)$. Show that if $V(I)$ has codimension[3] at least 2, then $V(I)$ is empty.
(3) Deduce that if X is an affine integral S_2 scheme then if $f \in \mathcal{O}_X(U)$ with $X \setminus U$ being of codimension at least 2, then $f \in \mathcal{O}_X(X)$.

Remark. We can drop the affine condition and do basically the same exercise using the language of quasi-coherent sheaves.

Exercise 4. *Fibers (2).*

(1) Show that for the morphism $\mathrm{Spec}(k[x,y]/(xy)) \to \mathrm{Spec}(k[x])$, induced by the obvious map $k[x] \to k[x,y]/(xy)$, every fiber is irreducible, although the $\mathrm{Spec}(k[x,y]/(xy))$ is not.
(2) Show that for the morphism $\mathrm{Spec}(\mathbb{Q}[t]) \to \mathrm{Spec}(\mathbb{Q}[t])$ induced by $t \mapsto t^2$ there are infinitely many closed points with irreducible fibers and infinitely many closed points with non-irreducible fibers.

[3]meaning that every irreducible component of $V(I)$ has codimension at least 2

Exercise 5. *Separated schemes.* Use the definition of separated maps to show the following.

(1) Show that a scheme X is separated if and only for every pair of open affines U and V, the intersection $U \cap V$ is affine and the natural map $\mathcal{O}_X(U) \otimes \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is surjective.
(2) Show that $\mathrm{Proj}(B) \to \mathrm{Spec}(\mathbb{Z})$ is separated for any $\mathbb{N}$-graded ring B. (Using (1) can be handy)
(3) Let k be a field. Let X be the scheme which is the gluing of two copies of $\mathbb{A}^1_k = \mathrm{Spec}(k[t])$ on the open subscheme $\mathbb{G}_m = \mathrm{Spec}(k[t, t^{-1}])$. Show that $X \to \mathrm{Spec}(k)$ is not separated.

Exercise 6. *Gradings, geometrically.* In what follows we write

$$\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[t, t^{-1}]).$$

Note that $\mathbb{Z}[t, t^{-1}] \to \mathbb{Z}[x, x^{-1}, y, y^{-1}]$ defined by $t \mapsto xy$ gives a map of schemes[4]

$$m\colon \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$$

which we call the *multiplication.*
A $\mathbb{G}_m$-action on an affine scheme $X = \mathrm{Spec}(A)$ is a map[5]

$$\mu\colon \mathbb{G}_m \times X \to X$$

such that the following diagram commutes[6]

$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m \times X & \xrightarrow{\mathrm{id} \times \mu} & \mathbb{G}_m \times X \\ {\scriptstyle m \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathbb{G}_m \times X & \xrightarrow{\quad \mu \quad} & X \end{array}$$

and if c denotes the map $c\colon X \to \mathbb{G}_m \times X$ induced by the evaluation a 1, then $\mu c = \mathrm{id}$.

(1) Show that the set of $\mathbb{Z}$-gradings on a ring A (meaning gradings that turn A into a graded ring) is in one to one correspondence with the set of $\mathbb{G}_m$-actions on $\mathrm{Spec}(A)$.
(2) Let $d > 1$, S and R two $\mathbb{Z}$-graded rings, with associated action μ_S and μ_R. Show that (where $(-)^d\colon \mathbb{G}_m \to \mathbb{G}_m$ is induced by Spec by the ring map determined by $t \mapsto t^d$)

$$\begin{array}{ccc} \mathbb{G}_m \times \mathrm{Spec}(S) & \xrightarrow{\mu_S} & \mathrm{Spec}(S) \\ {\scriptstyle ((-)^d, f)}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{G}_m \times \mathrm{Spec}(R) & \xrightarrow{\mu_R} & \mathrm{Spec}(R) \end{array}$$

commutes[7] if and only φ factors through $S^{(d)} = \bigoplus_n S_{nd}$, meaning that is it *homogeneous of degree d.* Compare with the notion introduced in Exercise 5, week 4.

[4]We have $\mathbb{G}_m \times \mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[x, x^{-1}, y, y^{-1}]$.
[5]We have $\mathbb{G}_m \times X = \mathrm{Spec}(A[t, t^{-1}]$.
[6]We have $\mathbb{G}_m \times \mathbb{G}_m \times X = \mathrm{Spec}(A[x, y, x^{-1}, y^{-1}]$.
[7]On points, this means that $f(\lambda x) = \lambda^d f(x)$.

(3) Let A be a $\mathbb{Z}$-graded ring and $A_0 \to A$ the inclusion. Show that $\pi\colon \operatorname{Spec}(A) \to \operatorname{Spec}(A_0)$ has the following universal property. For every affine scheme X and map $f\colon \operatorname{Spec}(A) \to X$ with the property that

$$\begin{array}{ccc} \mathbb{G}_m \times \operatorname{Spec}(A) & \xrightarrow{\mu} & \operatorname{Spec}(A) \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec}(A) & \xrightarrow{f} & X \end{array} \qquad (\mathbb{G}_m\text{-invariant maps})$$

then there exists a unique map $\overline{f}\colon \operatorname{Spec}(A_0) \to X$ with $\overline{f}\pi = f$. It means by definition that $\pi : \operatorname{Spec}(A) \to \operatorname{Spec}(A_0)$ is the quotient by the action of $\mathbb{G}_m$ in the category of affine schemes.